The Binomary Card Trick

A magician has six cards, each containing thirty-two numbers. All of the numbers are between 1 and 63. A volunteer from the audience is selected and asked to choose one of the numbers 1-63. While the magician turns his back the volunteer shows the audience which number she has chosen. The magician then asks the volunteer to hand him all of the cards that have her chosen number written on it. Immediately the magician reveals the chosen number.

Here are the magician's cards:

16 17 18 19 20 21 22 23
24 25 26 27 28 29 30 31
48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63

4 5 6 7 12 13 14 15
20 21 22 23 28 29 30 31
36 37 38 39 44 45 46 47
52 53 54 55 60 61 62 63

8 9 10 11 12 13 14 15
24 25 26 27 28 29 30 31
40 41 42 43 44 45 46 47
56 57 58 59 60 61 62 63

1 3 5 7 9 11 13 15
17 19 21 23 25 27 29 31
33 35 37 39 41 43 45 47
49 51 53 55 57 59 61 63

2 3 6 7 10 11 14 15
18 19 22 23 26 27 30 31
34 35 38 39 42 43 46 47
50 51 54 55 58 59 62 63

32 33 34 35 36 37 38 39
40 41 42 43 44 45 46 47
48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63

How does he do it? Do you notice anything special about the numbers on the cards?

How the Magician Does the Trick

Remember that a number between 1 and 63 has been (secretly) chosen, and that the magician is then handed those cards that contain the secret number. The magician then immediately reveals the secret number. How does he do it? He simply adds up the numbers that appear first (the upper-left-hand number) on the card.

As an example, suppose the number 51 is chosen. It appears on the cards starting with numbers 16,1,2, and 32. The sum of these numbers (1+2+16+32) is, of course, 51. Try this with a few more numbers and check that the sums always add to the original number.

16 17 18 19 20 21 22 23
24 25 26 27 28 29 30 31
48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63

4 5 6 7 12 13 14 15
20 21 22 23 28 29 30 31
36 37 38 39 44 45 46 47
52 53 54 55 60 61 62 63

8 9 10 11 12 13 14 15
24 25 26 27 28 29 30 31
40 41 42 43 44 45 46 47
56 57 58 59 60 61 62 63

1 3 5 7 9 11 13 15
17 19 21 23 25 27 29 31
33 35 37 39 41 43 45 47
49 51 53 55 57 59 61 63

2 3 6 7 10 11 14 15
18 19 22 23 26 27 30 31
34 35 38 39 42 43 46 47
50 51 54 55 58 59 62 63

32 33 34 35 36 37 38 39
40 41 42 43 44 45 46 47
48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63

Notice that all of the numbers appearing first on the cards are powers of 2: $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, etc, up to $32 = 2^5$. This is because we are using binary, or base-2, numbers to make up the cards.

About Binomial Numbers

Binomial, or base-2, numbers are numbers based on powers of two, just as our decimal, or base-10, numbers are based on powers of ten. Recall that the decimal number 6351 really means $6 \times 10^3 + 3 \times 10^2 + 5 \times 10 + 1$. Binary numbers work in exactly the same way, except that they use powers of two rather than ten. So the binary number 10011 means $1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + +1 \times 2^1 + 1 \times 1$, or 16+2+1 which is 19 as a decimal.

As another example we find the decimal equivalent of 101011. This as a decimal is $1 \times 2^5 + 1 \times 2^3 + 1 \times 2 + 1 \times 1 = 32 + 8 + 2 + 1 = 43$. You might want to try 11011 and show that its decimal equivalent is 27.

What does all this have to do with the magic trick? Take the number 19, as above, which is 10011 in binary. Remember that 10011 means $1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + +1 \times 2^1 + 1 \times 1 = 16 + 2 + 1$. If we look at the cards with first numbers 16,2,and 1 we see that these three cards are the ones containing the number 19. That is, the number 19 has been placed on exactly those cards starting with the powers of 2 corresponding to a 1 in the binary expansion. (And so those numbers must sum to the number.)

In the same way the number 27, which is 101011 in binary, or $1 \times 2^5 + 1 \times 2^3 + 1 \times 2 + 1 \times 1 = 32 + 8 + 2 + 1$, and you can check that it is on the cards starting with 1,2,8, and 32. The same holds for any other number between 1 and 63.

Exercise: Show that 1110 represents the number 14 and that 14 is on the cards beginning with 2, 4, and 8.

Finally, we might ask how the cards were created. To do this we need to be able to find the binary representation of a decimal number. This is easily shown with a couple of examples. Recall that the first few powers of 2 are

$$1, 2, 4, 8, 16, \text{ and } 32$$

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Number: 38. We start by writing 38 as the sum of powers of 2. The highest power of 2 that is less than 38 is 32, write 38=32+6. Since 6 is not a power of two, we find the biggest power of 2 less than 6, which is 4, so we now have 38=32+4+2. Since 2 is a power of two we are done. Adding in all the skipped powers of 2 gives, $38 = 1 \times 32 + 0 \times 16 + 0 \times 8 + 1 \times 4 + 1 \times 2 + 0 \times 1$, or the binary number 100110. We would place the number 38 on the cards with first number 32, 4 and 2.

Number: 15. Using the highest powers of two possible we find that 15=8+4+2+1, which is 1111 in binary. We would place the number 15 on those cards with first number 8,4,2, and 1.

To make up a set of binary magic-cards we need to find the binary expansion of each number 1-63, and then place the numbers on those cards beginning with a power of two corresponding to a 1 in the binary expansion of the numbers.

A Small Example

We will make up cards for the numbers 1-7. (Seven is $2^3 - 1$. Always choose one less than a power of two for nice cards.) These cards are too small to be interesting as a magic trick, but will show show the larger cards were created.

1=1
2=2
3=2+1 (So 3 goes on the cards beginning with 2 and 1)
4=4
5= 4+1 (5 goes on cards beginning 4 and 1)
6 = 4+2 (6 goes on cards beginning 4 and 2)
7=4+2+1 (7 goes on cards beginning 4,2 and 1)

We see that we will only need three cards, one beginning with 1, one with 2, and one 4. Making up these cards and following the instructions above as to where to put the other numbers we get the cards:

1 3 5 7

2 3 6 7

4 5 6 7

Thats all there is to it. We simply place each card where its binary expansions appear as the first numbers on the cards.